Generalized Einstein Aggregation Operators Based on the Interval Neutrosophic Numbers and Their Application to Multi-attribute Group Decision Making

Abstract: Based on the Einstein operator, the operational rules of interval neutrosophic numbers are defined, according to the combination of Einstein operations and generalized aggregation operators, the interval neutrosophic generalized weighted Einstein average (INGWEA) operator, interval neutrosophic generalized ordered weighted Einstein average (INGOWEA) operator and interval neutrosophic generalized hybrid weighted Einstein average (INGHWEA) operator are proposed . And some desirable properties of these operators including idempotency, monotonicity, commutativity and boundedness are studied, and some special cases including the interval neutrosophic numbers weighted Einstein averaging operators and the interval neutrosophic numbers weighted Einstein geometric operators are analyzed in the meantime. Furthermore, to address the multiple attribute group decision making (MAGDM) problems in which attribute values are interval neutrosophic numbers oriented, a group decision making method based on generalized Einstein aggregation operators is developed.

Keywords: interval neutrosophic number; Einstein operator; generalized averaging operator; multiple attribute group decision making

1. Introduction

Group decision theory and method have comprehensive application in many fields for instance political, economic, military, science, culture and so on. But it is easier to use linguistic information to evaluate alternatives and solve the group decision making problems for decision makers. Firstly, Zadeh [1] `s fuzzy set (FS) theory proposed in 1965 was a useful tool to deal with fuzzy information which has acquired rapid development and a wide range of application in fuzzy multi-attribute decision making (MADM) problems. However, the shortage of it is that it has no non-membership other than membership . Further Atanassov [2] `s intuitionistic fuzzy set (IFS) which added a non-membership function extend the FS, that is to say, the intuitionistic fuzzy sets can be expressed by $T_A(x)$ and $F_A(x)$ which means membership (or called truth-membership) and non-membership (or called falsity-membership) respectively. It has made contributions to multi-attribute decision making (MADM) and multi-attribute group decision making (MAGDM) [3-5]. Atanassov and Gargov [6], Atanassov [7] further developed the interval-valued intuitionistic fuzzy set (IVIFS) and defined some basic

operational rules, the most important is these rules extended both the truth-membership function and falsity- membership function to interval numbers. However, IFSs and IVIFSs, which can only deal with incomplete information and inconsistent information environment which exist commonly in real occasions. For example, during a voting process, that thirty percent vote “yes”, twenty percent vote “no”, ten percent give up and forty percent are undecided. It can not be presented by the IFS,because the indeterminacy ($1\text{-}T_A(x)\text{-}F_A(x)$) does not included in IFSs. Seeing that this situation, Smarandache [8] presented the concept of the neutrosophic set (NS) which is a generalization of IFS and have three parts: $T_A(x)$, $I_A(x)$ and $F_A(x)$ which mean truth-membership, indeterminacy-membership and falsity-membership respectively. Obviously, NS is useful to represent opinions more accurately and detailedly for decision makers. And we should know that the indeterminacy is quantified definitely and $T_A(x)$, $I_A(x)$ and $F_A(x)$ are independent absolutely in NS. Such as the case mentioned above, it can be showed as $x(0.3, 0.4, 0.2)$ from the point view of NS.

So far, a lot of researches on NSs mostly contain the following: The single valued neutrosophic set (SVNS) considered as a case of the neutrosophic set proposed by Wang et al. [9].The interval neutrosophic set (INS) in which $T_A(x)$, $I_A(x)$ and $F_A(x)$ were extended to interval numbers defined by Wang et al. [10].The correlation coefficient and weighted correlation coefficient of SVNSs developed by Ye [11] who also proved that the correlation coefficient in SVNS is the generalization of the cosine similarity degree. The similarity measures between INSs in accordance with the Hamming and Euclidean distances represented by Ye [12], then he put forward a multi-criteria decision-making method which is on the basis of the similarity degree. To this day, what draw people`s eyes is the characteristics of INS rather than aggregation operators for INS.

In addition, it is the information aggregation operators that have become an important research direction and got considerable research results. Einstein operators which considered as a special type of t-norm have been studied and applied to decision making by some researchers. Therefore,on the basis of Einstein operations, Wang and Liu [13] developed the intuitionistic fuzzy Einstein aggregation operators which the intuitionistic fuzzy Einstein weighted geometric (IFEWG) operator and the intuitionistic fuzzy Einstein ordered weighted geometric (IFEOWG) operator included, and have the better smooth approximations than the algebraic operators. Zhao and Wei [14] established intuitionistic fuzzy MADM methods based on intuitionistic fuzzy Einstein hybrid average (IFEHA) operator and intuitionistic fuzzy Einstein hybrid geometric (IFEHG) operator they developed. Guo et al. [15] proposed hesitant fuzzy Einstein weighted geometric (HFEWG) operator, hesitant fuzzy Einstein ordered weighted geometric (HFEOWG) operator, hesitant fuzzy Einstein hybrid geometric (HFEHG) operator, and hesitant fuzzy Einstein induced ordered weighted geometric (HFEIOWG) operator extended Einstein operators to hesitant fuzzy sets. Even so, there is nobody starting researches about

neutrosophic number aggregation operators according to Einstein operations.

Seeing that for one thing INS generalizes NS to interval numbers and can indicate indeterminacy more close to the actualities for decision makers, for another arithmetic aggregation operators and geometric aggregation operators are the special case of generalized aggregation operators. This paper tries to represent some generalized Einstein aggregation operators based on interval neutrosophic numbers by means of uniting both Einstein operators and generalized aggregation operators which include interval neutrosophic generalized weighted Einstein average (INGWEA) operator, interval neutrosophic generalized ordered weighted Einstein average (INGOWEA) operator and interval neutrosophic generalized hybrid weighted Einstein average (INGHWEA) operator. In the meantime, according to generalized parameter we discuss the properties and special cases of arithmetic aggregation operators and geometric aggregation operators. Ultimately, we propose a useful method for MAGDM problems in which attribute values are interval neutrosophic numbers oriented.

To accomplish the goal mentioned above, the remainder of this paper is arranged as follows. In section 2, some basic concepts such as INS, Einstein operations and generalized aggregation operators are briefly reviewed. In section 3, the new operational laws of INSs based on Einstein operators are established and the characteristics are discussed. In section 4, interval neutrosophic generalized weighted Einstein average (INGWEA) operator, interval neutrosophic generalized ordered weighted Einstein average (INGOWEA) operator and interval neutrosophic generalized hybrid weighted Einstein average (INGHWEA) operator which use new operational laws aforementioned in section 3 are developed, and some desirable properties and special cases are also discussed. In section 5, a decision making method on basis of INGHWEA operator for the MAGDM problems in which attribute values are interval neutrosophic numbers oriented. In section 6, an example to illustrate the application of proposed method is given, and we make a comparison between the developed method and the existing methods. Section 7 summarizes the main conclusion of this paper.

2. Preliminaries

2.1 The Interval Neutrosophic Set (INS)

Definition 1 [8]. Let X be a universe of discourse, with a generic element in X denoted by x. A neutrosophic set A in X is

$$A = \{x\langle T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\} \tag{1}$$

Where T_A is the truth-membership function, I_A is the indeterminacy-membership function, and F_A is the falsity-membership function. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or nonstandard

subsets of $]0^-,1^+[$ with not necessarily any connection between them.

There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \le T_A(x)+I_A(x)+F_A(x) \le 3^+$.

The neutrosophic set was originated from philosophy, which leaded it to apply to real life difficultly. The single valued neutrosophic set (SVNS) as a generalization of classical set, fuzzy set, intuitionistic fuzzy set and paraconsistent sets etc.put forward by Wang [9] which was represented as follows.

Definition 2 [9]. Let X be a universe of discourse, with a generic element in X denoted by x. A single valued neutrosophic set A in X is

$$A = \{x\langle T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\} \quad (2)$$

Where T_A is the truth-membership function, I_A is the indeterminacy-membership function, and F_A is the falsity-membership function. For each point x in X, we have $T_A(x), I_A(x), F_A(x) \in [0,1]$, and $0 \le T_A(x)+I_A(x)+F_A(x) \le 3$.

As for the real situations, it is difficult to express T_A, I_A and F_A using crisp numbers, and they can be expressed by interval numbers. Wang et al. [11] further defined interval neutrosophic sets (INSs) shown as follows.

Definition 3 [11]. Let X be a universe of discourse, with a generic element in X denoted by x. A interval neutrosophic set A in X is

$$A = \{x\langle T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\} \quad (3)$$

where, T_A is the truth-membership function, I_A is the indeterminacy-membership function, and F_A is the falsity-membership function. For each point x in X, we have $T_A(x), I_A(x), F_A(x) \subseteq [0,1]$, and $0 \le \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \le 3$.

For convenience, we can use $x=([T^L,T^U],[I^L,I^U],[F^L,F^U])$ which can be called an interval neutrosophic number (INN) to represent an element in INS.

Definition 4. Let $x=([T_1^L,T_1^U],[I_1^L,I_1^U],[F_1^L,F_1^U])$ and $y=([T_2^L,T_2^U],[I_2^L,I_2^U],[F_2^L,F_2^U])$ be any two INNs, then the cosine of included angle between x and y is

$$\cos(x,y)=\frac{T_1^LT_2^L+T_1^UT_2^U+I_1^LI_2^L+I_1^UI_2^U+F_1^LF_2^L+F_1^UF_2^U}{\sqrt{(T_1^L)^2+(T_1^U)^2+(I_1^L)^2+(I_1^U)^2+(F_1^L)^2+(F_1^U)^2}\bullet\sqrt{(T_2^L)^2+(T_2^U)^2+(I_2^L)^2+(I_2^U)^2+(F_2^L)^2+(F_2^U)^2}} \tag{4}$$

Obviously, $0<\cos(x,y)\le 1$. When y is the ideal solution I=([1,1],[0,0],[0,0]), the bigger the $\cos(x,I)$ between x and I is, the more consistent the direction between x and I is. In this condition,

$$\cos^*(x,I)=\frac{T_1^L+T_1^U}{\sqrt{2\left((T_1^L)^2+(T_1^U)^2+(I_1^L)^2+(I_1^U)^2+(F_1^L)^2+(F_1^U)^2\right)}} \tag{5}$$

Definition 5. Let $x=([T_1^L,T_1^U],[I_1^L,I_1^U],[F_1^L,F_1^U])$ and $y=([T_2^L,T_2^U],[I_2^L,I_2^U],[F_2^L,F_2^U])$ be any two INNs, if $\cos^*(x,I)<\cos^*(y,I)$ then $x<y$.

2.2 Generalized hybrid weighted aggregation operations

The generalized weighted average (GWA) operator is a generalization of the weighted average operator, which is defined as follows:

Definition 6 [16]**.** Let $GWA:(R^+)^n\to R^+$, if

$$GWA(a_1,a_2,\cdots,a_n)=\left(\sum_{j=1}^{n}w_ja_j^\lambda\right)^{1/\lambda} \tag{6}$$

where $W=(w_1,w_2,\cdots,w_n)^T$ is the weighting vector of the a_j $(j=1,2,\cdots,n)$ such that

$w_j \in [0,1], \sum_{j=1}^{n} w_j = 1$, and λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$, then we can call GWA the generalized weighted average operator.

Definition 7 [15]**.** Let $GOWA: (R^{+})^{n} \to R^{+}$, if

$$GOWA(a_1, a_2, \cdots, a_n) = \left(\sum_{j=1}^{n} \omega_j b_j^{\lambda} \right)^{1/\lambda} \tag{7}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weighting vector associated with $GOWA$, which satisfies $\omega_j \in [0,1]$, $\sum_{j=1}^{n} \omega_j = 1$, b_j is the j th largest of a_i $(i = 1, 2, \cdots, n)$ and λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$, then we call $GOWA$ the generalized ordered weighted average operator.

Definition 8 [16]**.** Let $GHWA: (R^{+})^{n} \to R^{+}$, if

$$GHWA(a_1, a_2, \cdots, a_n) = \left(\sum_{j=1}^{n} \omega_j b_j^{\lambda} \right)^{1/\lambda} \tag{8}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weighting vector associated with $GHWA$, which satisfies $\omega_j \in [0,1]$, $\sum_{j=1}^{n} \omega_j = 1$, b_j is the j th largest of $(nw_i a_i)$ $(i = 1, 2, \cdots, n)$, $w = (w_1, w_2, \cdots, w_n)^T$ is the weighting vector of $(a_1, a_2, \cdots, a_n)$ such that $w_j \in [0,1], \sum_{j=1}^{n} w_j = 1$, and λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$, then we can all $GHWA$ the generalized hybrid weighted average operator.

2.3 Einstein operations

In reality, the t-operators are Intersection and Union operators in fuzzy set theory which are symbolized by T-norm (T), and T-conorm (T^*), respectively. Smarandache and Vladareanu [18]

introduced a generalized union and a generalized intersection of single valued neutrosophic numbers on the basis of a T-norm and T-conorm .

Definition 9 [17]. Let $x=(T_1,I_1,F_1)$ and $y=(T_2,I_2,F_2)$ be any two single valued neutrosophic numbers, then, the generalized intersection and union are defined as follows:

$$x\bigcap_{\Gamma,\Gamma^*} y=\left(\Gamma(T_1,T_2),\Gamma^*(I_1,I_2),\Gamma^*(F_1,F_2)\right) \tag{9}$$

$$x\bigcup_{\Gamma,\Gamma^*} y=\left(\Gamma^*(T_1,T_2),\Gamma(I_1,I_2),\Gamma(F_1,F_2)\right) \tag{10}$$

where Γ denotes a T-norm and Γ^* a T-conorm.

Further, on the basis of T-norm and T-conorm, Einstein operations are defined as follows:

$$\Gamma_E(x,y)=\frac{xy}{1+(1-x)(1-y)} \tag{11}$$

$$\Gamma^*{}_E(x,y)=\frac{x+y}{1+xy} \tag{12}$$

where Γ denotes a T-norm and Γ^* a T-conorm.

Let $x=(T_1,I_1,F_1)$ and $y=(T_2,I_2,F_2)$ be any two single valued neutrosophic numbers, then, the operational rules of single valued neutrosophic numbers based on Einstein operations are as follows:

$$\text{(i)}\quad x\oplus_E y=\left(\frac{T_1+T_2}{1+T_1T_2},\frac{I_1I_2}{1+(1-I_1)(1-I_2)},\frac{F_1F_2}{1+(1-F_1)(1-F_2)}\right) \tag{13}$$

$$\text{(ii)}\quad x\otimes_E y=\left(\frac{T_1T_2}{1+(1-T_1)(1-T_2)},\frac{I_1+I_2}{1+I_1I_2},\frac{F_1+F_2}{1+F_1F_2}\right) \tag{14}$$

$$\text{(iii)}\quad \lambda x=\left(\frac{(1+T_1)^{\lambda}-(1-T_1)^{\lambda}}{(1+T_1)^{\lambda}+(1-T_1)^{\lambda}},\frac{2I_1^{\lambda}}{(2-I_1)^{\lambda}+I_1^{\lambda}},\frac{2F_1^{\lambda}}{(2-F_1)^{\lambda}+F_1^{\lambda}}\right),\lambda>0 \tag{15}$$

(iv) $$x^{\lambda}=\left(\frac{2T_1^{\lambda}}{\left(2-T_1\right)^{\lambda}+T_1^{\lambda}},\frac{\left(1+I_1\right)^{\lambda}-(1-I_1)^{\lambda}}{\left(1+I_1\right)^{\lambda}+(1-I_1)^{\lambda}},\frac{\left(1+F_1\right)^{\lambda}-(1-F_1)^{\lambda}}{\left(1+F_1\right)^{\lambda}+(1-F_1)^{\lambda}}\right),\lambda>0 \quad (16)$$

3. Einstein operations of interval neutrosophic numbers

For establishing the Einstein operations of INNs, we should give the following definition in the first place.

Definition 10. Let $x=([T_1^L,T_1^U],[I_1^L,I_1^U],[F_1^L,F_1^U])$ and $y=([T_2^L,T_2^U],[I_2^L,I_2^U],[F_2^L,F_2^U])$ be any two INNs, then, the generalized intersection and union are defined as follows:

$$x\otimes_{\Gamma,\Gamma^*}y=\Big([\Gamma(T_1^L,T_2^L),\Gamma(T_1^U,T_2^U)],[\Gamma^*(I_1^L,I_2^L),\Gamma^*(I_1^U,I_2^U)],\\ [\Gamma^*(F_1^L,F_2^L),\Gamma^*(F_1^U,F_2^U)]\Big) \quad (17)$$

$$x\oplus_{\Gamma,\Gamma^*}y=\Big([\Gamma^*(T_1^L,T_2^L),\Gamma^*(T_1^U,T_2^U)],[\Gamma(I_1^L,I_2^L),\Gamma(I_1^U,I_2^U)],\\ [\Gamma(F_1^L,F_2^L),\Gamma(F_1^U,F_2^U)]\Big) \quad (18)$$

where Γ denotes a T-norm and Γ^* a T-conorm.

According to definition 10, Einstein T-norm and T-conorm, the Einstein operational rules of INNs are established in the following.

Let $x=([T_1^L,T_1^U],[I_1^L,I_1^U],[F_1^L,F_1^U])$ and $y=([T_2^L,T_2^U],[I_2^L,I_2^U],[F_2^L,F_2^U])$ be any two INNs, then, the operational rules of INNs based on Einstein operations are shown as follows:

(1) $$x\oplus_E y=\left(\left[\frac{T_1^L+T_2^L}{1+T_1^LT_2^L},\frac{T_1^U+T_2^U}{1+T_1^UT_2^U}\right],\left[\frac{I_1^LI_2^L}{1+(1-I_1^L)(1-I_2^L)},\frac{I_1^UI_2^U}{1+(1-I_1^U)(1-I_2^U)}\right],\\ \left[\frac{F_1^LF_2^L}{1+(1-F_1^L)(1-F_2^L)},\frac{F_1^UF_2^U}{1+(1-F_1^U)(1-F_2^U)}\right]\right) \quad (19)$$

(2) $$x\otimes_E y=\left(\left[\frac{T_1^LT_2^L}{1+(1-T_1^L)(1-T_2^L)},\frac{T_1^UT_2^U}{1+(1-T_1^U)(1-T_2^U)}\right],\left[\frac{I_1^L+I_2^L}{1+I_1^LI_2^L},\frac{I_1^U+I_2^U}{1+I_1^UI_2^U}\right],\\ \left[\frac{F_1^L+F_2^L}{1+F_1^LF_2^L},\frac{F_1^U+F_2^U}{1+F_1^UF_2^U}\right]\right) \quad (20)$$

(3)
$$\lambda x=\left(\left[\frac{\left(1+T_1^L\right)^{\lambda}-(1-T_1^L)^{\lambda}}{\left(1+T_1^L\right)^{\lambda}+(1-T_1^L)^{\lambda}},\frac{\left(1+T_1^U\right)^{\lambda}-(1-T_1^U)^{\lambda}}{\left(1+T_1^U\right)^{\lambda}+(1-T_1^U)^{\lambda}}\right],\left[\frac{2I_1^{L\lambda}}{\left(2-I_1^L\right)^{\lambda}+I_1^{L\lambda}},\frac{2I_1^{U\lambda}}{\left(2-I_1^U\right)^{\lambda}+I_1^{U\lambda}}\right],\left[\frac{2F_1^{L\lambda}}{\left(2-F_1^L\right)^{\lambda}+F_1^{L\lambda}},\frac{2F_1^{U\lambda}}{\left(2-F_1^U\right)^{\lambda}+F_1^{U\lambda}}\right]\right),\lambda>0 \quad (21)$$

(4)
$$x^{\lambda}=\left(\left[\frac{2T_1^{L\lambda}}{\left(2-T_1^L\right)^{\lambda}+T_1^{L\lambda}},\frac{2T_1^{U\lambda}}{\left(2-T_1^U\right)^{\lambda}+T_1^{U\lambda}}\right],\left[\frac{\left(1+I_1^L\right)^{\lambda}-(1-I_1^L)^{\lambda}}{\left(1+I_1^L\right)^{\lambda}+(1-I_1^L)^{\lambda}},\frac{\left(1+I_1^U\right)^{\lambda}-(1-I_1^U)^{\lambda}}{\left(1+I_1^U\right)^{\lambda}+(1-I_1^U)^{\lambda}}\right],\left[\frac{\left(1+F_1^L\right)^{\lambda}-(1-F_1^L)^{\lambda}}{\left(1+F_1^L\right)^{\lambda}+(1-F_1^L)^{\lambda}},\frac{\left(1+F_1^U\right)^{\lambda}-(1-F_1^U)^{\lambda}}{\left(1+F_1^U\right)^{\lambda}+(1-F_1^U)^{\lambda}}\right]\right),\lambda>0 \quad (22)$$

Theorem 1. Let $x=([T_1^L,T_1^U],[I_1^L,I_1^U],[F_1^L,F_1^U])$ and $y=([T_2^L,T_2^U],[I_2^L,I_2^U],[F_2^L,F_2^U])$ be any two INNs, then

(1) $x\oplus_E y=y\oplus_E x$ (23)

(2) $x\otimes_E y=y\otimes_E x$ (24)

(3) $\eta(x\oplus_E y)=\eta x\oplus_E \eta y,\eta\geq 0$ (25)

(4) $\eta_1 x\oplus_E \eta_2 x=(\eta_1+\eta_2)x,\ \ \eta_1,\eta_2\ \geq 0$ (26)

(5) $x^{\eta_1}\otimes_E x^{\eta_2}=x^{\eta_1+\eta_2},\ \ \eta_1,\eta_2\ \geq 0$ (27)

(6) $x^{\eta}\otimes_E y^{\eta}=(x\otimes_E y)^{\eta},\ \ \eta\geq 0$ (28)

It is easy to prove the above characteristics in theorem 1, so omitted in here.

4. Interval neutrosophic generalized weighted Einstein aggregation operators

Definition 11. Let $x_j = ([T_j^L, T_j^U],[I_j^L, I_j^U],[F_j^L, F_j^U])$ $(j = 1, 2, \cdots, n)$ be a collection of the INNs, and $INGWEA : \Omega^n \to \Omega$, if

$$INGWEA(x_1, x_2, \cdots, x_n) = \left(\bigoplus_{E\, j=1}^{n} \left(w_j x_j^{\lambda} \right) \right)^{1/\lambda} \tag{29}$$

where Ω is the set of all INNs, $w = (w_1, w_2, \cdots, w_n)^T$ is the weighting vector of the $(x_1, x_2, \cdots, x_n)$ such that $w_j \in [0,1], \sum_{j=1}^{n} w_j = 1$, and λ is a parameter such that $\lambda \in (0, +\infty)$, then we can call $INGWEA$ the interval neutrosophic generalized weighted Einstein average operator.

Theorem 2. Let $x_j = ([T_j^L, T_j^U],[I_j^L, I_j^U],[F_j^L, F_j^U])$ $(j = 1, 2, \cdots, n)$ be a collection of the INNs, then, the result aggregated from Definition 11 is still an INN, and even

$$INGWEA(x_1, x_2, \cdots, x_n) =$$

$$\left(\left[\frac{2\left(\prod_{j=1}^{n} A_{T_j^L}^{w_j} - \prod_{j=1}^{n} B_{T_j^L}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} A_{T_j^L}^{w_j} + 3\prod_{j=1}^{n} B_{T_j^L}^{w_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} A_{T_j^L}^{w_j} - \prod_{j=1}^{n} B_{T_j^L}^{w_j}\right)^{1/\lambda}}, \frac{2\left(\prod_{j=1}^{n} A_{T_j^U}^{w_j} - \prod_{j=1}^{n} B_{T_j^U}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} A_{T_j^U}^{w_j} + 3\prod_{j=1}^{n} B_{T_j^U}^{w_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} A_{T_j^U}^{w_j} - \prod_{j=1}^{n} B_{T_j^U}^{w_j}\right)^{1/\lambda}} \right], \right.$$

$$\left[\frac{\left(\prod_{j=1}^{n} C_{I_j^L}^{w_j} + 3\prod_{j=1}^{n} D_{I_j^L}^{w_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} C_{I_j^L}^{w_j} - \prod_{j=1}^{n} D_{I_j^L}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} C_{I_j^L}^{w_j} + 3\prod_{j=1}^{n} D_{I_j^L}^{w_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} C_{I_j^L}^{w_j} - \prod_{j=1}^{n} D_{I_j^L}^{w_j}\right)^{1/\lambda}}, \frac{\left(\prod_{j=1}^{n} C_{I_j^U}^{w_j} + 3\prod_{j=1}^{n} D_{I_j^U}^{w_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} C_{I_j^U}^{w_j} - \prod_{j=1}^{n} D_{I_j^U}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} C_{I_j^U}^{w_j} + 3\prod_{j=1}^{n} D_{I_j^U}^{w_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} C_{I_j^U}^{w_j} - \prod_{j=1}^{n} D_{I_j^U}^{w_j}\right)^{1/\lambda}} \right],$$

$$\left. \left[\frac{\left(\prod_{j=1}^{n} M_{F_j^L}^{w_j} + 3\prod_{j=1}^{n} N_{F_j^L}^{w_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} M_{F_j^L}^{w_j} - \prod_{j=1}^{n} N_{F_j^L}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} M_{F_j^L}^{w_j} + 3\prod_{j=1}^{n} N_{F_j^L}^{w_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} M_{F_j^L}^{w_j} - \prod_{j=1}^{n} N_{F_j^L}^{w_j}\right)^{1/\lambda}}, \frac{\left(\prod_{j=1}^{n} M_{F_j^U}^{w_j} + 3\prod_{j=1}^{n} N_{F_j^U}^{w_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} M_{F_j^U}^{w_j} - \prod_{j=1}^{n} N_{F_j^U}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} M_{F_j^U}^{w_j} + 3\prod_{j=1}^{n} N_{F_j^U}^{w_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} M_{F_j^U}^{w_j} - \prod_{j=1}^{n} N_{F_j^U}^{w_j}\right)^{1/\lambda}} \right] \right)$$

(30)

where $A_{T_j^L} = \left(2-T_j^L\right)^{\lambda} + 3(T_j^L)^{\lambda}$, $B_{T_j^L} = \left(2-T_j^L\right)^{\lambda} - (T_j^L)^{\lambda}$, $A_{T_j^U} = \left(2-T_j^U\right)^{\lambda} + 3(T_j^U)^{\lambda}$, $B_{T_j^U} = \left(2-T_j^U\right)^{\lambda} - (T_j^U)^{\lambda}$, $C_{I_j^L} = \left(1+I_j^L\right)^{\lambda} + 3(1-I_j^L)^{\lambda}$, $D_{I_j^L} = \left(1+I_j^L\right)^{\lambda} - (1-I_j^L)^{\lambda}$, $C_{I_j^U} = (1+I_j^U)^{\lambda} + 3(1-I_j^U)^{\lambda}$, $D_{I_j^U} = (1+I_j^U)^{\lambda} - (1-I_j^U)^{\lambda}$, $M_{F_j^L} = \left(1+F_j^L\right)^{\lambda} + 3(1-F_j^L)^{\lambda}$, $N_{F_j^L} = \left(1+F_j^L\right)^{\lambda} - (1-F_j^L)^{\lambda}$, $M_{F_j^U} = (1+F_j^U)^{\lambda} + 3(1-F_j^U)^{\lambda}$, $N_{F_j^U} = (1+F_j^U)^{\lambda} - (1-F_j^U)^{\lambda}$.

Because of the constraints of space, the proof of Theorem 2 is omitted here.

It is easy to prove that the $INGWEA$ operator has the following three properties.

(1) Theorem 3 (Monotonicity)

Let $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ be any two collections of the INNs, if $x_j \le y_j$ for all j, then $INGWEA(x_1, x_2, \cdots, x_n) \le INGWEA(y_1, y_2, \cdots, y_n)$.

(2) Theorem 4 (Idempotency)

Let $x_j = x = ([T^L, T^U], [I^L, I^U], [F^L, F^U]),\ j = 1, 2, \cdots, n$, then $INGWEA(x_1, x_2, \cdots, x_n) = x$.

(3) Theorem 5 (Boundedness)

The $INGWEA$ operator lies between the max and min operators: $\min(x_1, x_2, \cdots, x_n) \le INGWEA(x_1, x_2, \cdots, x_n) \le \max(x_1, x_2, \cdots, x_n)$

We will discuss some special cases of $INGWEA$ operator according to the value of λ as follows.

(1) When $\lambda = 1$

$$INGWEA_{\lambda=1}(x_1,x_2,\cdots,x_n)=\bigoplus_{E\, j=1}^{n}\left(w_j x_j\right)$$

$$=\left(\left[\frac{\prod_{j=1}^{n}\left(1+T_j^L\right)^{w_j}-\prod_{j=1}^{n}(1-T_j^L)^{w_j}}{\prod_{j=1}^{n}\left(1+T_j^L\right)^{w_j}+\prod_{j=1}^{n}(1-T_j^L)^{w_j}},\frac{\prod_{j=1}^{n}\left(1+T_j^U\right)^{w_j}-\prod_{j=1}^{n}(1-T_j^U)^{w_j}}{\prod_{j=1}^{n}\left(1+T_j^U\right)^{w_j}+\prod_{j=1}^{n}(1-T_j^U)^{w_j}}\right],\right.$$
$$\left[\frac{2\prod_{j=1}^{n}(I_j^L)^{w_j}}{\prod_{j=1}^{n}\left(2-I_j^L\right)^{w_j}+\prod_{j=1}^{n}(I_j^L)^{w_j}},\frac{2\prod_{j=1}^{n}(I_j^U)^{w_j}}{\prod_{j=1}^{n}\left(2-I_j^U\right)^{w_j}+\prod_{j=1}^{n}(I_j^U)^{w_j}}\right], \tag{31}$$
$$\left.\left[\frac{2\prod_{j=1}^{n}(F_j^L)^{w_j}}{\prod_{j=1}^{n}\left(2-F_j^L\right)^{w_j}+\prod_{j=1}^{n}(F_j^L)^{w_j}},\frac{2\prod_{j=1}^{n}(F_j^U)^{w_j}}{\prod_{j=1}^{n}\left(2-F_j^U\right)^{w_j}+\prod_{j=1}^{n}(F_j^U)^{w_j}}\right]\right)$$

The $INGWEA$ operator degenerates into interval neutrosophic Einstein weighted average $(INEWA)$ operator.

(2) When $\lambda\to 0$

$$INGWEA_{\lambda\to 0}(x_1,x_2,\cdots,x_n)=\bigotimes_{E\, j=1}^{n}x_j^{w_j}$$

$$=\left(\left[\frac{2\prod_{j=1}^{n}(T_j^L)^{w_j}}{\prod_{j=1}^{n}\left(2-T_j^L\right)^{w_j}+\prod_{j=1}^{n}(T_j^L)^{w_j}},\frac{2\prod_{j=1}^{n}(T_j^U)^{w_j}}{\prod_{j=1}^{n}\left(2-T_j^U\right)^{w_j}+\prod_{j=1}^{n}(T_j^U)^{w_j}}\right],\right.$$
$$\left[\frac{\prod_{j=1}^{n}\left(1+I_j^L\right)^{w_j}-\prod_{j=1}^{n}(1-I_j^L)^{w_j}}{\prod_{j=1}^{n}\left(1+I_j^L\right)^{w_j}+\prod_{j=1}^{n}(1-I_j^L)^{w_j}},\frac{\prod_{j=1}^{n}\left(1+I_j^U\right)^{w_j}-\prod_{j=1}^{n}(1-I_j^U)^{w_j}}{\prod_{j=1}^{n}\left(1+I_j^U\right)^{w_j}+\prod_{j=1}^{n}(1-I_j^U)^{w_j}}\right], \tag{32}$$
$$\left.\left[\frac{\prod_{j=1}^{n}\left(1+F_j^L\right)^{w_j}-\prod_{j=1}^{n}(1-F_j^L)^{w_j}}{\prod_{j=1}^{n}\left(1+F_j^L\right)^{w_j}+\prod_{j=1}^{n}(1-F_j^L)^{w_j}},\frac{\prod_{j=1}^{n}\left(1+F_j^U\right)^{w_j}-\prod_{j=1}^{n}(1-F_j^U)^{w_j}}{\prod_{j=1}^{n}\left(1+F_j^U\right)^{w_j}+\prod_{j=1}^{n}(1-F_j^U)^{w_j}}\right]\right)$$

The $INGWEA$ operator reduces to interval neutrosophic Einstein weighted geometric $(INEWG)$ operator.

Definition 12. Let $x_j = ([T_j^L, T_j^U],[I_j^L, I_j^U],[F_j^L, F_j^U])$ $(j = 1, 2, \cdots, n)$ be a collection of the INNs, and $INGOWEA: \Omega^n \to \Omega$, if

$$INGOWEA(x_1, x_2, \cdots, x_n) = \left(\bigoplus_{E\,j=1}^{n} \left(\omega_j x_{\sigma(j)}^{\lambda} \right) \right)^{1/\lambda} \tag{33}$$

where Ω is the set of all INNs, $\omega = \left(\omega_1, \omega_2, \cdots, \omega_n\right)^T$ is the weighting vector associated with $INGOWEA$, such that $\omega_j \in [0,1]$, $\sum_{j=1}^{n} \omega_j = 1$. $\left(\sigma(1), \sigma(2), \cdots, \sigma(n)\right)$ is a permutation of $(1, 2, \cdots, n)$ which satisfies $x_{\sigma(j-1)} \geq x_{\sigma(j)}$ for all j, and λ is a parameter such that $\lambda \in (0, +\infty)$, then we can call $INGOWEA$ the interval neutrosophic generalized ordered weighted Einstein average operator.

Theorem 6. Let $x_j = ([T_j^L, T_j^U],[I_j^L, I_j^U],[F_j^L, F_j^U])$ $(j = 1, 2, \cdots, n)$ be a collection of the INNs and $\lambda > 0$, then, the result aggregated from Definition 12 is still an INN, and even

$$
\begin{aligned}
& INGOWEA(x_1, x_2, \cdots, x_n) \\
&= \left(\left[\frac{2\left(\prod_{j=1}^{n} A_{T^L_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} B_{T^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} A_{T^L_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} B_{T^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} A_{T^L_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} B_{T^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}, \right.\right. \\
&\quad \left. \frac{2\left(\prod_{j=1}^{n} A_{T^U_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} B_{T^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} A_{T^U_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} B_{T^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} A_{T^U_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} B_{T^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}} \right], \\
&\quad \left[\frac{\left(\prod_{j=1}^{n} C_{I^L_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} D_{I^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} C_{I^L_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} D_{I^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} C_{I^L_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} D_{I^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} C_{I^L_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} D_{I^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}, \right. \\
&\quad \left. \frac{\left(\prod_{j=1}^{n} C_{I^U_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} D_{I^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} C_{I^U_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} D_{I^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} C_{I^U_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} D_{I^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} C_{I^U_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} D_{I^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}} \right], \\
&\quad \left[\frac{\left(\prod_{j=1}^{n} M_{F^L_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} N_{F^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} M_{F^L_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} N_{F^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} M_{F^L_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} N_{F^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} M_{F^L_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} N_{F^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}, \right. \\
&\quad \left.\left. \frac{\left(\prod_{j=1}^{n} M_{F^U_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} N_{F^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} M_{F^U_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} N_{F^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} M_{F^U_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} N_{F^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} M_{F^U_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} N_{F^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}} \right]\right)
\end{aligned}
\tag{34}
$$

where $A_{T_j^L} = \left(2 - T_j^L\right)^{\lambda} + 3(T_j^L)^{\lambda}$, $B_{T_j^L} = \left(2 - T_j^L\right)^{\lambda} - (T_j^L)^{\lambda}$, $A_{T_j^U} = \left(2 - T_j^U\right)^{\lambda} + 3(T_j^U)^{\lambda}$, $B_{T_j^U} = \left(2 - T_j^U\right)^{\lambda} - (T_j^U)^{\lambda}$, $C_{I_j^L} = \left(1 + I_j^L\right)^{\lambda} + 3(1 - I_j^L)^{\lambda}$, $D_{I_j^L} = \left(1 + I_j^L\right)^{\lambda} - (1 - I_j^L)^{\lambda}$, $C_{I_j^U} = (1 + I_j^U)^{\lambda} + 3(1 - I_j^U)^{\lambda}$, $D_{I_j^U} = (1 + I_j^U)^{\lambda} - (1 - I_j^U)^{\lambda}$, $M_{F_j^L} = \left(1 + F_j^L\right)^{\lambda} + 3(1 - F_j^L)^{\lambda}$, $N_{F_j^L} = \left(1 + F_j^L\right)^{\lambda} - (1 - F_j^L)^{\lambda}$, $M_{F_j^U} = (1 + F_j^U)^{\lambda} + 3(1 - F_j^U)^{\lambda}$,

$$N_{F_j^U} = (1+F_j^U)^{\lambda} - (1-F_j^U)^{\lambda}.$$

Limited to the space, the proof of Theorem 6 is omitted here.

Because the input arguments are weighted according to the arguments position in descending order for the whole arguments in the $INGOWEA$ operator, so the $\omega = \left(\omega_1, \omega_2, \cdots, \omega_n\right)^T$ is called the position weighted vector. Position weighted vector $\omega = \left(\omega_1, \omega_2, \cdots, \omega_n\right)^T$ should be assigned according to the real decision-making needs generally.

It is easy to prove that the $INGOWEA$ operator has the four properties.

(1) Theorem 7 (Monotonicity)

Let $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ be any two collections of the INNs, if $x_j \le y_j$ for all j, then $INGOWEA(x_1, x_2, \cdots, x_n) \le INGOWEA(y_1, y_2, \cdots, y_n)$.

(2) Theorem 8 (Idempotency)

Let $x_j = x = ([T^L, T^U], [I^L, I^U], [F^L, F^U]),\ j = 1, 2, \cdots, n$, then $INGOWEA(x_1, x_2, \cdots, x_n) = x$.

(3) Theorem 9 (Boundedness)

The $INGWEA$ operator lies between the max and min operators: $\min(x_1, x_2, \cdots, x_n) \le INGOWEA(x_1, x_2, \cdots, x_n) \le \max(x_1, x_2, \cdots, x_n)$

(4) Theorem10 (Commutativity)

Let $(x_1, x_2, \cdots, x_n)$ be any permutation of $(y_1, y_2, \cdots, y_n)$, then $INGOWEA(x_1, x_2, \cdots, x_n) = INGOWEA(y_1, y_2, \cdots, y_n)$.

Similarly, as for different values of parameter λ, some special cases of $INGOWEA$ operator will be discussed in the following.

(1) When $\lambda = 1$

$$INGOWEA_{\lambda=1}(x_1,x_2,\cdots,x_n)=\bigoplus_{E\,j=1}^{n}\left(\omega_j x_{\sigma(j)}\right)$$

$$=\left(\begin{array}{l}\left[\dfrac{\prod_{j=1}^{n}\left(1+T_{\sigma(j)}^{L}\right)^{\omega_j}-\prod_{j=1}^{n}(1-T_{\sigma(j)}^{L})^{\omega_j}}{\prod_{j=1}^{n}\left(1+T_{\sigma(j)}^{L}\right)^{\omega_j}+\prod_{j=1}^{n}(1-T_{\sigma(j)}^{L})^{\omega_j}},\dfrac{\prod_{j=1}^{n}\left(1+T_{\sigma(j)}^{U}\right)^{\omega_j}-\prod_{j=1}^{n}(1-T_{\sigma(j)}^{U})^{\omega_j}}{\prod_{j=1}^{n}\left(1+T_{\sigma(j)}^{U}\right)^{\omega_j}+\prod_{j=1}^{n}(1-T_{\sigma(j)}^{U})^{\omega_j}}\right],\\ \left[\dfrac{2\prod_{j=1}^{n}(I_{\sigma(j)}^{L})^{\omega_j}}{\prod_{j=1}^{n}\left(2-I_{\sigma(j)}^{L}\right)^{\omega_j}+\prod_{j=1}^{n}(I_{\sigma(j)}^{L})^{\omega_j}},\dfrac{2\prod_{j=1}^{n}(I_{\sigma(j)}^{U})^{\omega_j}}{\prod_{j=1}^{n}\left(2-I_{\sigma(j)}^{U}\right)^{\omega_j}+\prod_{j=1}^{n}(I_{\sigma(j)}^{U})^{\omega_j}}\right],\\ \left[\dfrac{2\prod_{j=1}^{n}(F_{\sigma(j)}^{L})^{\omega_j}}{\prod_{j=1}^{n}\left(2-F_{\sigma(j)}^{L}\right)^{\omega_j}+\prod_{j=1}^{n}(F_{\sigma(j)}^{L})^{\omega_j}},\dfrac{2\prod_{j=1}^{n}(F_{\sigma(j)}^{U})^{\omega_j}}{\prod_{j=1}^{n}\left(2-F_{\sigma(j)}^{U}\right)^{\omega_j}+\prod_{j=1}^{n}(F_{\sigma(j)}^{U})^{\omega_j}}\right]\end{array}\right) \tag{35}$$

The $INGOWEA$ operator degenerates into interval neutrosophic Einstein ordered weighted average $(INEOWA)$ operator.

(2) When $\lambda\to 0$

$$INGOWEA_{\lambda\to 0}(x_1,x_2,\cdots,x_n)=\bigotimes_{E\,j=1}^{n}x_{\sigma(j)}^{\omega_j}$$

$$=\left(\begin{array}{l}\left[\dfrac{2\prod_{j=1}^{n}(T_{\sigma(j)}^{L})^{\omega_j}}{\prod_{j=1}^{n}\left(2-T_{\sigma(j)}^{L}\right)^{\omega_j}+\prod_{j=1}^{n}(T_{\sigma(j)}^{L})^{\omega_j}},\dfrac{2\prod_{j=1}^{n}(T_{\sigma(j)}^{U})^{\omega_j}}{\prod_{j=1}^{n}\left(2-T_{\sigma(j)}^{U}\right)^{\omega_j}+\prod_{j=1}^{n}(T_{\sigma(j)}^{U})^{\omega_j}}\right],\\ \left[\dfrac{\prod_{j=1}^{n}\left(1+I_{\sigma(j)}^{L}\right)^{\omega_j}-\prod_{j=1}^{n}(1-I_{\sigma(j)}^{L})^{\omega_j}}{\prod_{j=1}^{n}\left(1+I_{\sigma(j)}^{L}\right)^{\omega_j}+\prod_{j=1}^{n}(1-I_{\sigma(j)}^{L})^{\omega_j}},\dfrac{\prod_{j=1}^{n}\left(1+I_{\sigma(j)}^{U}\right)^{\omega_j}-\prod_{j=1}^{n}(1-I_{\sigma(j)}^{U})^{\omega_j}}{\prod_{j=1}^{n}\left(1+I_{\sigma(j)}^{U}\right)^{\omega_j}+\prod_{j=1}^{n}(1-I_{\sigma(j)}^{U})^{\omega_j}}\right],\\ \left[\dfrac{\prod_{j=1}^{n}\left(1+F_{\sigma(j)}^{L}\right)^{\omega_j}-\prod_{j=1}^{n}(1-F_{\sigma(j)}^{L})^{\omega_j}}{\prod_{j=1}^{n}\left(1+F_{\sigma(j)}^{L}\right)^{\omega_j}+\prod_{j=1}^{n}(1-F_{\sigma(j)}^{L})^{\omega_j}},\dfrac{\prod_{j=1}^{n}\left(1+F_{\sigma(j)}^{U}\right)^{\omega_j}-\prod_{j=1}^{n}(1-F_{\sigma(j)}^{U})^{\omega_j}}{\prod_{j=1}^{n}\left(1+F_{\sigma(j)}^{U}\right)^{\omega_j}+\prod_{j=1}^{n}(1-F_{\sigma(j)}^{U})^{\omega_j}}\right]\end{array}\right) \tag{36}$$

The $INGOWEA$ operator reduces to interval neutrosophic Einstein ordered weighted geometric

$(INEOWG)$ operator.

What we discussed the two interval neutrosophic Einstein operators, including both $INGWEA$ operator and $INGOWEA$ operator. One hand the $INGWEA$ operator lays stress on their own importance of these arguments, while on the other hand $INGOWEA$ operator emphasizes their ordered position importance. However, as for real decision making problems, we need take consider the two weighted modes into consideration which indicate different aspects of decision-making problems. In order to overcome the weakness that each operator considers only one aspect, generalized hybrid averaging operator based on Einstein operations is proposed as follows.

Definition 13. Let $x_j = ([T_j^L, T_j^U],[I_j^L, I_j^U],[F_j^L, F_j^U])$ $(j = 1, 2, \cdots, n)$ be a collection of the INNs, and $INGHWEA : \Omega^n \to \Omega$, if

$$INGHWEA(x_1, x_2, \cdots, x_n) = \left(\bigoplus_{E\, j=1}^{n} \left(\omega_j s_{\sigma(j)}^{\lambda} \right) \right)^{1/\lambda} \tag{37}$$

where Ω is the set of all INNs, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weighting vector associated with $INGHWEA$, such that $\omega_j \in [0,1], \sum_{j=1}^{n} \omega_j = 1$. $w = (w_1, w_2, \cdots, w_n)^T$ is the weighting vector of the $(x_1, x_2, \cdots, x_n)$ such that $w_j \in [0,1], \sum_{j=1}^{n} w_j = 1$. Let $s_j = n w_j x_j = ([\dot{T}_j^L, \dot{T}_j^U],[\dot{I}_j^L, \dot{I}_j^U],[\dot{F}_j^L, \dot{F}_j^U])$, n is the adjustment factor. $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$ which satisfies $x_{\sigma(j-1)} \geq x_{\sigma(j)}$ for all j, and λ is a parameter such that $\lambda \in (0, +\infty)$, then we can call $INGHWEA$ the interval neutrosophic generalized hybrid weighted Einstein average operator.

Theorem 11. Let $x_j = ([T_j^L, T_j^U],[I_j^L, I_j^U],[F_j^L, F_j^U])$ $(j = 1, 2, \cdots, n)$ be a collection of the INNs and $\lambda > 0$, then, the result aggregated from Definition 13 is still an INN, and even

$$
\begin{aligned}
& INGHWEA(x_1, x_2, \cdots, x_n) \\
&= \left(\left[\frac{2\left(\prod_{j=1}^{n} \dot{A}_{T^L_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} \dot{B}_{T^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} \dot{A}_{T^L_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} \dot{B}_{T^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} \dot{A}_{T^L_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} \dot{B}_{T^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}, \right.\right. \\
&\qquad \left. \frac{2\left(\prod_{j=1}^{n} \dot{A}_{T^U_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} \dot{B}_{T^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} \dot{A}_{T^U_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} \dot{B}_{T^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} \dot{A}_{T^U_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} \dot{B}_{T^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}} \right], \\
&\qquad \left[\frac{\left(\prod_{j=1}^{n} \dot{C}_{I^L_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} \dot{D}_{I^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} \dot{C}_{I^L_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} \dot{D}_{I^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} \dot{C}_{I^L_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} \dot{D}_{I^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} \dot{C}_{I^L_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} \dot{D}_{I^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}, \right. \\
&\qquad \left. \frac{\left(\prod_{j=1}^{n} \dot{C}_{I^U_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} \dot{D}_{I^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} \dot{C}_{I^U_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} \dot{D}_{I^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} \dot{C}_{I^U_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} \dot{D}_{I^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} \dot{C}_{I^U_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} \dot{D}_{I^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}} \right], \\
&\qquad \left[\frac{\left(\prod_{j=1}^{n} \dot{M}_{F^L_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} \dot{N}_{F^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} \dot{M}_{F^L_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} \dot{N}_{F^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} \dot{M}_{F^L_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} \dot{N}_{F^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} \dot{M}_{F^L_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} \dot{N}_{F^L_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}, \right. \\
&\qquad \left.\left. \frac{\left(\prod_{j=1}^{n} \dot{M}_{F^U_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} \dot{N}_{F^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} \dot{M}_{F^U_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} \dot{N}_{F^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} \dot{M}_{F^U_{\sigma(j)}}^{\omega_j} + 3\prod_{j=1}^{n} \dot{N}_{F^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} \dot{M}_{F^U_{\sigma(j)}}^{\omega_j} - \prod_{j=1}^{n} \dot{N}_{F^U_{\sigma(j)}}^{\omega_j}\right)^{1/\lambda}} \right] \right)
\end{aligned}
\tag{38}
$$

where $\dot{A}_{T_j^L} = \left(2 - \dot{T}_j^L\right)^{\lambda} + 3(\dot{T}_j^L)^{\lambda}$, $\dot{B}_{T_j^L} = \left(2 - \dot{T}_j^L\right)^{\lambda} - (\dot{T}_j^L)^{\lambda}$, $\dot{A}_{T_j^U} = \left(2 - \dot{T}_j^U\right)^{\lambda} + 3(\dot{T}_j^U)^{\lambda}$, $\dot{B}_{T_j^U} = \left(2 - \dot{T}_j^U\right)^{\lambda} - (\dot{T}_j^U)^{\lambda}$, $\dot{C}_{I_j^L} = \left(1 + \dot{I}_j^L\right)^{\lambda} + 3(1 - \dot{I}_j^L)^{\lambda}$, $\dot{D}_{I_j^L} = \left(1 + \dot{I}_j^L\right)^{\lambda} - (1 - \dot{I}_j^L)^{\lambda}$, $\dot{C}_{I_j^U} = (1 + \dot{I}_j^U)^{\lambda} + 3(1 - \dot{I}_j^U)^{\lambda}$, $\dot{D}_{I_j^U} = (1 + \dot{I}_j^U)^{\lambda} - (1 - \dot{I}_j^U)^{\lambda}$, $\dot{M}_{F_j^L} = \left(1 + \dot{F}_j^L\right)^{\lambda} + 3(1 - \dot{F}_j^L)^{\lambda}$, $\dot{N}_{F_j^L} = \left(1 + \dot{F}_j^L\right)^{\lambda} - (1 - \dot{F}_j^L)^{\lambda}$, $\dot{M}_{F_j^U} = (1 + \dot{F}_j^U)^{\lambda} + 3(1 - \dot{F}_j^U)^{\lambda}$,

$$\dot{N}_{F_j^U}=(1+\dot{F}_j^U)^{\lambda}-(1-\dot{F}_j^U)^{\lambda},\ \dot{T}_j^L=\frac{\left(1+T_j^L\right)^{nw_j}-(1-T_j^L)^{nw_j}}{\left(1+T_j^L\right)^{nw_j}+(1-T_j^L)^{nw_j}},\ \dot{T}_j^U=\frac{\left(1+T_j^U\right)^{nw_j}-(1-T_j^U)^{nw_j}}{\left(1+T_j^U\right)^{nw_j}+(1-T_j^U)^{nw_j}},$$

$$\dot{I}_j^L=\frac{2(I_j^L)^{nw_j}}{\left(2-I_j^L\right)^{nw_j}+(I_j^L)^{nw_j}},\ \dot{I}_j^U=\frac{2(I_j^U)^{nw_j}}{\left(2-I_j^U\right)^{nw_j}+(I_j^U)^{nw_j}},\ \dot{F}_j^L=\frac{2(F_j^L)^{nw_j}}{\left(2-F_j^L\right)^{nw_j}+(F_j^L)^{nw_j}},$$

$$\dot{F}_j^U=\frac{2(F_j^U)^{nw_j}}{\left(2-F_j^U\right)^{nw_j}+(F_j^U)^{nw_j}}.$$

Limited to the space, the proof of Theorem 11 is omitted here.

Theorem 12. The $INGWEA$ operator and $INGOWEA$ operator are the special cases of the $INGHWEA$ operator.

(1) If $\omega=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$, then $INGHWEA$ operator reduces to $INGWEA$ operator.

(2) If $W=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$, then $INGHWEA$ operator reduces to $INGOWEA$ operator.

Similarly, when $\lambda=1$, the $INGHWEA$ operator will degenerate into interval neutrosophic Einstein hybrid weighted average $(INEHWA)$ operator, and when $\lambda\to 0$, the $INGHWEA$ operator will degenerate into interval neutrosophic Einstein hybrid weighted geometric $(INEHWG)$ operator.

According to definition 13, we can know that the $INGHWEA$ operator firstly weights the given input arguments, and then reorders the weighted values in descending order and weights these ordered arguments.Therefore, the $INGHWEA$ operator can reflect the importance degrees of both the given input arguments and their ordered positions.

5. Multiple attribute group decision making method based on INGHWEA operator

Taking the MAGDM problems based on INNs into account: let $A=\{A_1,A_2,\cdots,A_m\}$,

$C=\left\{C_1, C_2, \cdots, C_n\right\}$ be the set of alternatives and attributes respectively, and w_j is the weight of the attribute $C_j\,(j=1,2,\cdots,n)$, where $0 \le w_j \le 1\ \ (j=1,2,\cdots,n)$, $\sum_{j=1}^{n} w_j = 1$. Suppose that $D=\left\{D_1, D_2, \cdots, D_t\right\}$ is the set of decision makers, and $\gamma_k\ (k=1,2,\cdots,t)$ is a weight of decision maker D_k with $0 \le \gamma_k \le 1\ (k=1,2,\cdots,t)$, $\sum_{k=1}^{t}\gamma_k = 1$. Suppose that $X^{(k)}=\left(x_{ij}^{(k)}\right)_{m\times n}$ is the decision matrix, where $x_{ij}^{(k)}=([T_{ij}^{L(k)}, T_{ij}^{U(k)}],[I_{ij}^{L(k)}, I_{ij}^{U(k)}],[F_{ij}^{L(k)}, F_{ij}^{U(k)}])$ takes the form of the INN and $[T_{ij}^{L(k)}, T_{ij}^{U(k)}],[I_{ij}^{L(k)}, I_{ij}^{U(k)}],[F_{ij}^{L(k)}, F_{ij}^{U(k)}]\subseteq[0,1]$, $0 \le sup(T_{ij}^{L(k)}, T_{ij}^{U(k)})+sup(I_{ij}^{L(k)}, I_{ij}^{U(k)})+sup(F_{ij}^{L(k)}, F_{ij}^{U(k)}) \le 3$, which means that the decision maker D_k evaluates the attribute C_j with respect to the alternative A_i. Then, we can rank the order of the alternatives based on the given information.

Attributes can be classified into two types including benefit attributes and cost attributes generally, and the n attributes may be evaluated in different ways, so what we should do is normalizing the decision-making information before applying the $INGHWEA$ operator to the MAGDM problems.

The method involves the following steps:

Step 1. Normalize the decision-making information.

Suppose $R^{(k)}=\left(r_{ij}^{(k)}\right)_{m\times n}\ (k=1,2,\cdots,t)$ is the normalized matrix of the decision matrix $X^{(k)}=\left(x_{ij}^{(k)}\right)_{m\times n}\ (k=1,2,\cdots,t)$, where $r_{ij}^{(k)}=([\underline{T}_{ij}^{(k)}, \overline{T}_{ij}^{(k)}],[\underline{I}_{ij}^{(k)}, \overline{I}_{ij}^{(k)}],[\underline{F}_{ij}^{(k)}, \overline{F}_{ij}^{(k)}])$, then the normalization method is chosen as follows:

(1) For benefit attributes:

$$r_{ij}^{(k)}=([\underline{T}_{ij}^{(k)}, \overline{T}_{ij}^{(k)}],[\underline{I}_{ij}^{(k)}, \overline{I}_{ij}^{(k)}],[\underline{F}_{ij}^{(k)}, \overline{F}_{ij}^{(k)}])=([T_{ij}^{L(k)}, T_{ij}^{U(k)}],[I_{ij}^{L(k)}, I_{ij}^{U(k)}],[F_{ij}^{L(k)}, F_{ij}^{U(k)}]) \qquad (39)$$

where $i=1,2,\cdots,m,\ \ j=1,2,\cdots,n,\ \ k=1,2,\cdots,t$.

(2) For cost attributes:

$$r_{ij}^{(k)}=([\underline{T}_{ij}^{(k)}, \overline{T}_{ij}^{(k)}],[\underline{I}_{ij}^{(k)}, \overline{I}_{ij}^{(k)}],[\underline{F}_{ij}^{(k)}, \overline{F}_{ij}^{(k)}])=([F_{ij}^{L(k)}, F_{ij}^{U(k)}],[1-I_{ij}^{U(k)}, 1-I_{ij}^{L(k)}],[T_{ij}^{L(k)}, T_{ij}^{U(k)}]) \qquad (40)$$

where $i=1,2,\cdots,m,\ \ j=1,2,\cdots,n,\ \ k=1,2,\cdots,t$.

Step 2. Aggregate the assessment information of each decision maker D_k by $INGHWEA$ operator, and get the collective information.

$$r_{ij} = ([\underline{T}_{ij}, \overline{T}_{ij}], [\underline{I}_{ij}, \overline{I}_{ij}], [\underline{F}_{ij}, \overline{F}_{ij}]) = INGHWEA(r_{ij}^{(1)}, r_{ij}^{(2)}, \cdots, r_{ij}^{(t)}) \tag{41}$$

Step 3. Calculate the comprehensive evaluation value of each alternative.

$$r_i = ([\underline{T}_i, \overline{T}_i], [\underline{I}_i, \overline{I}_i], [\underline{F}_i, \overline{F}_i]) = INGHWEA(r_{i1}, r_{i2}, \cdots, r_{in}) \tag{42}$$

Step 4. Calculate the cosine of included angle between r_i $(i = 1, 2, \cdots, m)$ and the ideal solution $I = ([1,1],[0,0],[0,0])$.

$$\cos^*(r_i, I) = \frac{\underline{T}_i + \overline{T}_i}{\sqrt{2\left(\left(\underline{T}_i\right)^2 + \left(\overline{T}_i\right)^2 + \left(\underline{I}_i\right)^2 + \left(\overline{I}_i\right)^2 + \left(\underline{F}_i\right)^2 + \left(\overline{F}_i\right)^2\right)}} \tag{43}$$

Step 5. Rank the alternatives $\{A_1, A_2, \cdots, A_m\}$ in descending order according to the Definition 5.

Step 6. End.

6. An illustrate Example

Suppose that the example about the air quality evaluation of Guangzhou for the 16th Asian Olympic Games (adapted from [18]). The air quality in Guangzhou for the Novembers of 2006, 2007, 2008 and 2009 were collected as the set of alternatives that is $\{A_1, A_2, A_3, A_4\}$ = {November of 2006, November of 2007, November of 2008, November of 2009}. There are three air-quality monitoring stations expressed by (D_1, D_2, D_3) which can be seen as decision makers, and their weight $\gamma = (0.314, 0.355, 0.331)^T$. The measured attributes (suppose that their weight vector is $w = (0.40, 0.20, 0.40)^T$) are shown as follows: SO_2 (C_1), NO_2 (C_2), PM_{10} (C_3).

The measured values from air-quality monitoring stations under these attributes are shown in tables 1, 2 and 3, and they can be expressed by INNs (Note: the original data take the form of interval intuitionistic fuzzy numbers, we can get INNs by $I^L = 1 - T^U - F^U$ and $I^U = 1 - T^L - F^L$).

Table 1 Air quality data from station D_1

	C_1	C_2	C_3
A_1	([0.22, 0.31],[0.15,0.55],[0.23, 0.54])	([0.13, 0.53],[0.11,0.67],[0.20, 0.36])	([0.12, 0.37],[0.07,0.48],[0.40, 0.56])
A_2	([0.28, 0.41],[0.10,0.39],[0.33, 0.49])	([0.33, 0.53],[0.11,0.47],[0.20, 0.36])	([0.12, 0.37],[0.17,0.58],[0.30, 0.46])
A_3	([0.32, 0.41],[0.15,0.45],[0.23, 0.44])	([0.43, 0.53],[0.22,0.41],[0.16, 0.25])	([0.23, 0.45],[0.18,0.56],[0.21, 0.37])
A_4	([0.39, 0.47],[0.17,0.43],[0.18, 0.36])	([0.39, 0.53],[0.15,0.34],[0.27, 0.32])	([0.28, 0.34],[0.43,0.61],[0.11, 0.23])

Table 2 Air quality data from station D_2

	C_1	C_2	C_3
A_1	([0.04, 0.21],[0.33,0.61],[0.35, 0.46])	([0.10, 0.34],[0.21,0.63],[0.27, 0.45])	([0.32, 0.37],[0,43,0.55],[0.13, 0.20])
A_2	([0.32, 0.39],[0.22,0.41],[0.27, 0.39])	([0.03, 0.57],[0.07,0.67],[0.30, 0.36])	([0.16, 0.25],[0.56,0.70],[0.14, 0.19])
A_3	([0.26, 0.37],[0.23,0.53],[0.21, 0.40])	([0.23, 0.43],[0.42,0.71],[0.06, 0.15])	([0.21, 0.35],[0.36,0.68],[0.11, 0.29])
A_4	([0.30, 0.43],[0.22,0.51],[0.19, 0.35])	([0.28, 0.43],[0.23,0.41],[0.31, 0.34])	([0.39, 0.46],[0.37,0.60],[0.01, 0.17])

Table 3 Air quality data from station D_3

	C_1	C_2	C_3
A_1	([0.25, 0.27],[0.33,0.52],[0.23, 0.40])	([0.17, 0.27],[0.33,0.57],[0.26, 0.40])	([0.21, 0.30],[0.38,0.62],[0.17, 0.32])
A_2	([0.25, 0.29],[0.32,0.42],[0.33, 0.39])	([0.18, 0.46],[0.04,0.39],[0.43, 0.50])	([0.06, 0.21],[0.49,0.66],[0.28, 0.30])

	C_1	C_2	C_3
A_3	([0.22, 0.27],[0.42,0.51],[0.27, 0.31])	([0.13, 0.37],[0.43,0.71],[0.16, 0.20])	([0.11, 0.24],[0.57,0.75],[0.14, 0.19])
A_4	([0.30, 0.48],[0.07,0.61],[0.09, 0.45])	([0.08, 0.53],[0.23,0.72],[0.20, 0.24])	([0.32, 0.61],[0.30,0.67],[0.01, 0.09])

6.1 The evaluation steps

The steps are shown as follows:

(1) Normalize the decision-making information.

Because the whole measured values are of the same type, so they do not need normalization.

(2) Aggregate the measured values of each air-quality monitoring station D_k by $INGHWEA$ operator in formula (41), suppose $\lambda = 2$ and $\omega = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Then we can get an aggregated matrix R based on three decision matrices measured by air-quality monitoring stations, which is shown as follows:

$$R=\begin{pmatrix} ([0.192,0.264],[0.259,0.560],[0.267,0.459]) & ([0.136,0.396],[0.200,0.618],[0.243,0.404]) & ([0.244,0.350],[0.237,0.546],[0.202,0.316]) \\ ([0.288,0.369],[0.196,0.407],[0.307,0.417]) & ([0.215,0.526],[0.067,0.495],[0.297,0.399]) & ([0.124,0.283],[0.365,0.646],[0.224,0.289]) \\ ([0.269,0.356],[0.245,0.497],[0.235,0.377]) & ([0.293,0.448],[0.345,0.582],[0.114,0.194]) & ([0.192,0.358],[0.333,0.655],[0.146,0.271]) \\ ([0.331,0.460],[0.140,0.510],[0.146,0.383]) & ([0.284,0.497],[0.201,0.451],[0.257,0.297]) & ([0.340,0.494],[0.360,0.624],[0.022,0.152]) \end{pmatrix}$$

(3) Calculate the comprehensive evaluation value of each alternative by formula (42), suppose $\lambda = 2$ and $\omega = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

$$r_1 = ([0.223, 0.334],[0.234, 0.545],[0.229, 0.372]) \quad r_2 = ([0.234, 0.373],[0.206, 0.490],[0.261, 0.343])$$
$$r_3 = ([0.252, 0.382],[0.287, 0.553],[0.168, 0.288]) \quad r_4 = ([0.344, 0.494],[0.219, 0.532],[0.077, 0.246])$$

(4) Calculate the cosine of included angle between r_i $(i = 1, 2, \cdots, m)$ and the ideal solution $I = ([1,1],[0,0],[0,0])$ based on formula (43).

$$\cos^*(r_1, I) = 0.469, \quad \cos^*(r_2, I) = 0.528, \quad \cos^*(r_3, I) = 0.532, \quad \cos^*(r_4, I) = 0.680$$

(5) Rank the alternatives

According to the comparison method described in Definition 5, the ranking is $A_4 \succ A_3 \succ A_2 \succ A_1$, So the best alternative is A_4, that is, the best air quality in Guangzhou is November of 2009 among the Novembers of 2006, 2007, 2008, and 2009.

6.2 Discussion

For demonstrating the influence of the parameter λ on decision making of this example, we change the value of λ in the second and third step to rank the alternatives, then get the ranking results are shown in Table 4.

Table 4 Ordering of the alternatives by utilizing the different λ in $INGHWEA$ operator

λ	$\cos^*(r_i, I)$	Ranking result
$\lambda = 0.01$	$\cos^*(r_1, I) = 0.398$ $\cos^*(r_2, I) = 0.460$ $\cos^*(r_3, I) = 0.486$ $\cos^*(r_4, I) = 0.637$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$\lambda = 0.5$	$\cos^*(r_1, I) = 0.399$ $\cos^*(r_2, I) = 0.460$ $\cos^*(r_3, I) = 0.485$ $\cos^*(r_4, I) = 0.636$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$\lambda = 1.0$	$\cos^*(r_1, I) = 0.417$ $\cos^*(r_2, I) = 0.478$ $\cos^*(r_3, I) = 0.496$ $\cos^*(r_4, I) = 0.647$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$\lambda = 1.5$	$\cos^*(r_1, I) = 0.443$ $\cos^*(r_2, I) = 0.502$ $\cos^*(r_3, I) = 0.513$ $\cos^*(r_4, I) = 0.663$	$A_4 \succ A_3 \succ A_2 \succ A_1$

$\lambda = 2.0$	$\cos^*(r_1, I) = 0.469$ $\cos^*(r_2, I) = 0.528$ $\cos^*(r_3, I) = 0.532$ $\cos^*(r_4, I) = 0.680$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$\lambda = 2.3$	$\cos^*(r_1, I) = 0.484$ $\cos^*(r_2, I) = 0.5425$ $\cos^*(r_3, I) = 0.5435$ $\cos^*(r_4, I) = 0.6895$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$\lambda = 2.4$	$\cos^*(r_1, I) = 0.489$ $\cos^*(r_2, I) = 0.5474$ $\cos^*(r_3, I) = 0.5471$ $\cos^*(r_4, I) = 0.693$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$\lambda = 5$	$\cos^*(r_1, I) = 0.587$ $\cos^*(r_2, I) = 0.650$ $\cos^*(r_3, I) = 0.620$ $\cos^*(r_4, I) = 0.757$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$\lambda = 10$	$\cos^*(r_1, I) = 0.678$ $\cos^*(r_2, I) = 0.747$ $\cos^*(r_3, I) = 0.688$ $\cos^*(r_4, I) = 0.818$	$A_4 \succ A_2 \succ A_3 \succ A_1$

As we can see from Table 4, the ordering of the alternatives may be different for the different value λ in $INGHWEA$ operator.

(1) When $0 < \lambda \leq 2.3$, the ordering of the alternatives is $A_4 \succ A_3 \succ A_2 \succ A_1$ and the best alternative is A_4.

(2) When $\lambda \geq 2.4$, the ordering of the alternatives is $A_4 \succ A_2 \succ A_3 \succ A_1$ and the best alternative is A_4.

From above analysis, we find that the ordering of the alternatives changes with the value of parameter λ. To some extent, λ can be considered as the mentality of the decision-makers, the more

the λ is, the more optimistic decision-makers are, and vice versa. Generally speaking, we consider the rankings of some special values when λ gets 0 and 1 as what we want. We can think the best alternative is A_4 in this example.

For confirming the effective of the proposed method, we utilized the method proposed by Yue [18] to solve this illustrate example, and we can get the ranking result $A_4 \succ A_3 \succ A_2 \succ A_1$. Because the proposed method has a variety of weighted average modes with the change of parameter λ, such as the geometric average, arithmetic average, quadratic average, and so on. When $0 < \lambda \leq 2.3$, the ordering of the alternatives is $A_4 \succ A_3 \succ A_2 \succ A_1$, which is the same as the ranking result by using the method proposed by Yue [18]. However, the method proposed in this paper become more flexible because of parameter λ.

7. Conclusion

During the process of solving the real problems, the decision maker always encounters the evaluation information of alternatives which is incomplete,indeterminate and inconsistent. Fortunately, the interval neutrosophic set (INS) is a better tool to depict this kind of information. In addition, Einstein operations meet the typical t-norm and t-conorm and have better smooth approximations than the algebraic operators. Therefore, in this paper, considering interval neutrosophic fuzzy information and on the basis of Einstein operators and generalized aggregation operators , interval neutrosophic generalized weighted Einstein average (INGWEA) operator, interval neutrosophic generalized ordered weighted Einstein average (INGOWEA) operator and interval neutrosophic generalized hybrid weighted Einstein average (INGHWEA) operator are proposed . Meanwhile, some properties of these operators are explored and special cases are discussed with respect to generalized parameter. Furthermore, we have given a decision making method based on INGHWEA operator.

This paper further extends Einstein operators to interval neutrosophic fuzzy information, and establishes a group decision-making method based on Einstein operators, which not only enriches neutrosophic set theory, but also further expands the using scope of Einstein operators. However, the proposed method only solves the MAGDM problems where the attribute values are INNs and the weights of attributes and decision makers are real numbers. In further research, we should study the MAGDM problems where the weights of attributes and decision makers are INNs, which exist commonly in real situations. In addition, it is necessary and meaningful to extend the applications of the operator to the other domains such as pattern recognition, fuzzy cluster analysis and uncertain programming, etc.

References

[1] L.A. Zadeh, Fuzzy sets, Information and Control 1965; 8(3): 338- 356.

[2] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 1986; 20(1): 87-96.

[3] D.F. Li, The GOWA operator based approach to multi-attribute decision making using intuitionistic fuzzy sets, Mathematical and Computer Modelling 2011; 53(5-6):1182–1196.

[4] Z.S. Xu, R.R. Yager, Intuitionistic Fuzzy Bonferroni Means, IEEE Transactions on Systems, Man, and Cybernetics—Part B: Cybernetics, 2011; 41(2):568-578.

[5] G.W. Wei, Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making, Applied Soft Computing 2010; 10(2) : 423–431.

[6] K.T. Atanassov, G. Gargov, Interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 1989; 31(3):343-349.

[7] K.T. Atanassov, Operators over interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 1994; 64(2): 159-174.

[8] F. Smarandache, A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth , 1999.

[9] H. Wang, F. Smarandache, Y. Zhang, R. Sunderraman, Single valued neutrosophic sets, Proc Of 10th 476 Int Conf on Fuzzy Theory and Technology, Salt Lake City, 477 Utah, 2005.

[10] J. Ye, Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment, International Journal of General Systems 2013; 42(4): 386-394.

[11] H. Wang, F. Smarandache, Y.Q. Zhang, et al., Interval neutrosophic sets and logic: Theory and applications in computing, Hexis, Phoenix, AZ, 2005.

[12] J. Ye, Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making, Journal of Intelligent & Fuzzy Systems 2014; 26(1): 165-172.

[13] W.Z. Wang, X.W. Liu, Intuitionistic fuzzy geometric aggregation operators based on Einstein operations, International Journal of Intelligent Systems 2011; 26(11): 1049-1075.

[14] X.F. Zhao, G.W. Wei, Some intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute decision making, Knowledge-Based Systems 2013; 37(1):472-479.

[15] H. Zhao, Z.S. Xu, M.F. Ni, S.S. Liu, Generalized aggregation operators for intuitionistic fuzzy sets, International Journal of Intelligent Systems 2010; 25(1): 1–30.

[16] S. Guo, F.F. Jin, H.Y. Chen, Hesitant fuzzy Einstein geometric aggregation operators and their

application, Computer Engineering and Applications 2013; doi:10.3778/j.issn.1002-8331.1309-0284.

[17] F. Smarandache, L. Vladareanu, Applications of Neutrosophic Logic to Robotics-An Introduction, 2011 IEEE International Conference on Granular Computing 2011; pp.607-612.

[18] Z.L. Yue, Deriving decision maker's weights based on distance measure for interval-valued intuitionistic fuzzy group decision making, Expert Systems with Applications 2011; 38 (9): 11665–11670.